Probability

CSCI 4511/6511

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Announcements

- Homework 3 Due 14 Oct
- Midterm Exam - 16 Oct
    - In class
    - Open note

Review

Symbols

- Propositional symbols
  - Similar to boolean variables
  - Either True or False

Sentences

- What is a linguistic sentence?
  - Subject(s)
  - Verb(s)
  - Object(s)
  - *Relationships*
- What is a logical sentence?
  - Symbols
  - Relationships

Familiar Logical Operators

- $\neg$
  - "Not" operator, same as CS (`!`, `not`, etc.)
- $\wedge$
  - "And" operator, same as CS (`&&`, `and`, etc.)
  - This is sometimes called a *conjunction*.
- $\vee$
  - "Inclusive Or" operator, same as CS.
  - This is sometimes called a *disjunction*.

Unfamiliar Logical Operators

- $\Rightarrow$
  - Logical *implication*.
  - If $X_0 \Rightarrow X_1$, X_1 is always True when X_0 is True.
  - If X_0 is False, the value of X_1 is not constrained.
- $\iff$
  - "If and only If."
  - If $X_0 \iff X_1$, X_0 and X_1 are either both True or both False.
  - Also called a *biconditional*.

Equivalent Statements

- $X_0 \Rightarrow X_1$ alternatively:
  - $(X_0 \wedge X_1) \vee \neg X_0$
- $X_0 \iff X_1$ alternatively:
  - $(X_0 \wedge X_1) \vee (\neg X_0 \wedge \neg X_1)$

Entailment

- $KB \models A$
  - "Knowledge Base entails A"
  - For every model in which KB is True, A is also True
  - One-way relationship: A can be True for models where KB is not True.
- Vocabulary: A is the *query*

Knowing Things

Falsehood:

- $KB \models \neg A$
  - No model exists where KB is True and A is True

It is possible to not know things:[1]

- $KB \not\models A$
- $KB \not\models \neg A$

1. $\not\models$ – “does not entail”

Satisfiability

- Commonly abbreviated “SAT”
- *First* NP-complete problem
- $(X_0 \wedge X_1) \vee X_2$
  - Satisfied by $X_0 = \text{True}, X_1 = \text{False}, X_2 = \text{True}$
  - Satisfied for any X_0 and X_1 if $X_2 = \text{True}$
- $X_0 \wedge \neg X_0 \wedge X_1$
  - Cannot be satisfied by any values of X_0 and X_1

Conjunctive Normal Form

- *Literals* — symbols or negated symbols
  - X_0 is a literal
  - $\neg X_0$ is a literal
- *Clauses* — combine literals and disjunction using disjunctions ($\vee$)
  - $X_0 \vee \neg X_1$ is a valid disjunction
  - $(X_0 \vee \neg X_1) \vee X_2$ is a valid disjunction

Conjunctive Normal Form

- *Conjunctions* ($\wedge$) combine clauses (and literals)
  - $X_1 \wedge (X_0 \vee \neg X_2)$
- Disjunctions cannot contain conjunctions:
- $X_0 \vee (X_1 \wedge X_2)$ not in CNF
  - Can be rewritten in CNF: $(X_0 \vee X_1) \wedge (X_0 \vee X_2)$

Converting to CNF

- $X_0 \iff X_1$
  - $(X_0 \Rightarrow X_1) \land (X_1 \Rightarrow X_0)$
- $X_0 \Rightarrow X_1$
  - $\neg X_0 \lor X_1$
- $\neg(X_0 \land X_1)$
  - $\neg X_0 \lor \neg X_1$
- $\neg(X_0 \lor X_1)$
  - $\neg X_0 \land \neg X_1$

Probability

Randomness and Uncertainty

- We don’t know things about future events
  - Someone else might know
- Example: expectimax!
  - Ghost could behave randomly
  - Ghost could behave according to some plan
  - We model behavior as random

The Random Variable

- Uncertain future event: random variable
- Probability:

$$P(x) = \lim_{n \to \infty} \frac{n_x}{n}$$

- Probabilities constrained $0 \leq P(x) \leq 1$ for any x

The Random Variable

- In ensemble of events, what fraction represent event x ?
  - What's troubling about this?
- How do we quantify probability based on observations?
- How do we quantify probability without direct observations?

Plausibility of Statements

- “A is more plausible than B”
  - $P(A) > P(B)$
- “A is as plausible as B”
  - $P(A) = P(B)$
- “A is impossible”
  - $P(A) = 0$
- “A is certain”
  - $P(A) = 1$

Probability Distribution

- Enumerate possible outcomes[1]
- Assign probabilities to outcomes
- Distribution: ensemble of outcomes mapped to probabilities
- Works for discrete and continuous cases

1. Everyone has them.

Combinatorics

- Enumerating outcomes is a counting problem
  - We know how to solve counting problems
- Permutations:
  - Ordering n items: $n!$
  - Ordering n items, k of which are alike: $\frac{n!}{k!}$
  - … k_1, k_2 of which are alike: $\frac{n!}{k_1!k_2!}$

I Am Extremely Sorry

…if you thought this course was going to be about LLMs

Combinatorics

- How many possible outcomes are there?
- How many possible outcomes are there *of interest?*
- Assume all outcomes have equal probability
  - Or don’t
- Divide
  - Weight if necessary

Choice

- n events
- k are of interest
  - $n - k$ are *not* of interest

Possible combinations:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Bernoulli Trials

- "Single event" that occurs with probability θ
- $P(E) = \theta$
- $P(\neg E) = 1 - \theta$
- Alternate notations:[1]
  - $P(E^C) = 1 - \theta$
  - $P(\bar{E}) = 1 - \theta$
- Examples?

1. Math notation can be inconsistent, which you may find infuriating.

Math Notation

- $P(E)$
  - Probability of some event E occuring
- $P\{X = a\}$
  - Probability of random variable X taking value a
- $p(a)$
  - Probability of random variable taking value a

Bernoulli Random Variable

- Bernoulli trial:
  - Variable, takes one of two values
  - Coin toss: H or T
  - $P\{X = H\} = \theta$
  - $P\{X = T\} = 1 - \theta$

Expected Value

- Variable's values can be numeric values:
  - Coin toss $H = 8$ and $T2$
  - $P\{X = 8\} = \theta$
  - $P\{X = 2\} = 1 - \theta$
- Expected value:
  - $E[X] = H \cdot \theta + T \cdot (1 - \theta)$
  - $E[X] = 8 \cdot \theta + 2 \cdot (1 - \theta)$

Expected Value

Of a variable:

$$E[X] = \sum_{i=0}^{n} x_i \cdot p(x_i)$$

Of a function of a variable:

$$E[g(x)] = \sum_{i=0}^{n} g(x_i) \cdot p(x_i) \neq g(E[X])$$

Variance

- How much do values *vary* from the expected value?

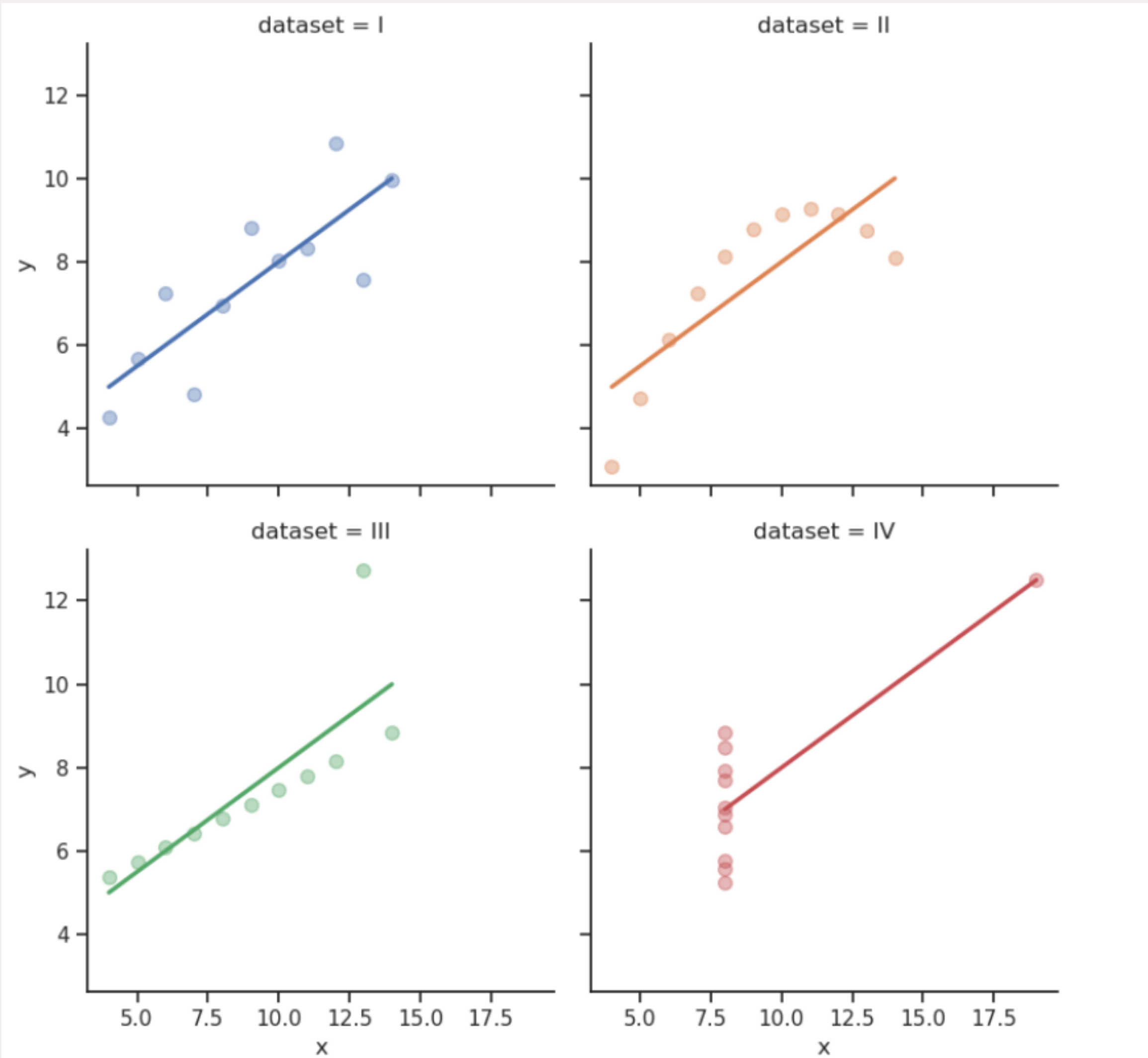
$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

- $E[X]$ represents mean, or μ
- We're really interested in $E[|X - \mu|]$
    - Absolute values are mathematically troublesome
- Standard deviation: $\sigma = \sqrt{\mathrm{Var}}$

Variance

$$
\begin{aligned}
\mathrm{Var}(X) &= E[(E[X] - \mu)^2] \\
&= \sum_x (x - \mu)^2 p(x) \\
&= \sum_x (x^2 - 2x\mu + \mu^2) p(x) \\
&= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) \\
&= E[X^2] - 2\mu\mu + \mu^2 \\
&= E[X^2] - E[X]^2
\end{aligned}
$$

How To Lie With Statistics



Discrete Distributions

Binomial Distribution

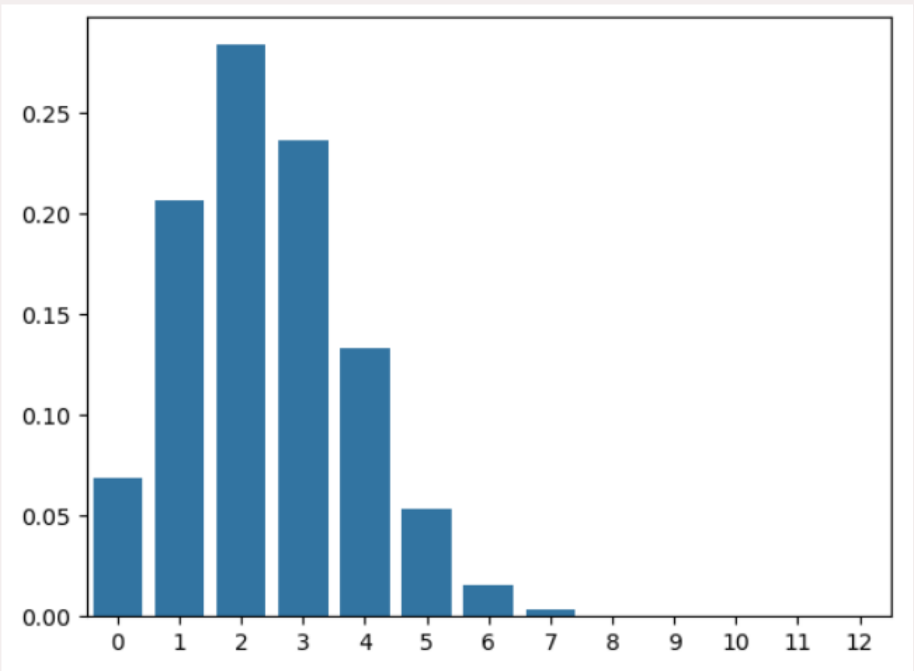
- Bernoulli trial:
  - Successes and failures
  - $P\{X = 1\} = \theta$
  - $P\{X = 0\} = 1 - \theta$
- Conduct *many* trials. How many succeed?

Binomial Distribution

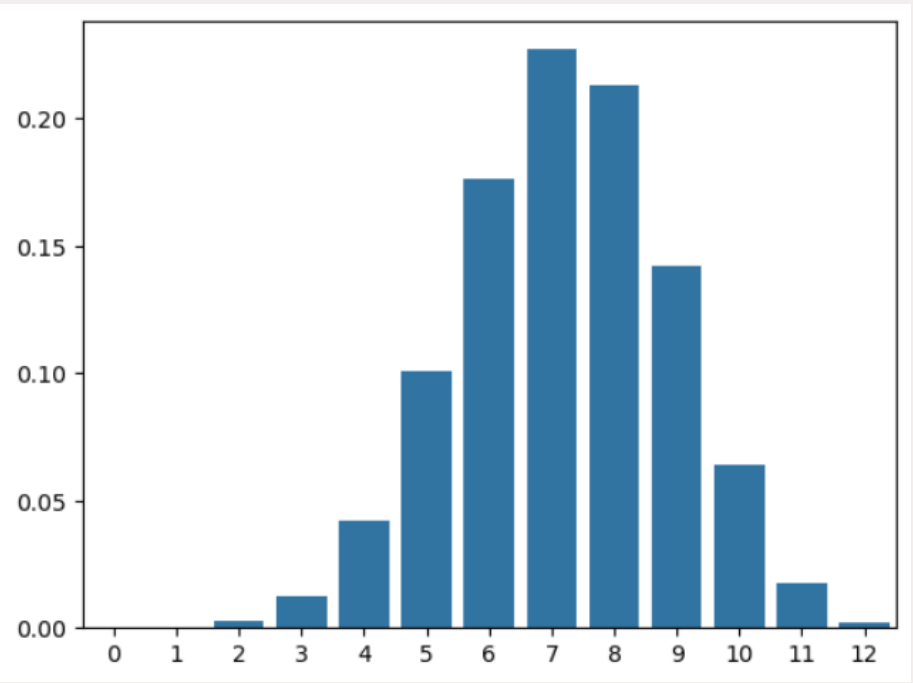
- Probability of n successes in n trials: θ^n
- Probability of k successes in n trials:
  - $\theta^k(1-\theta)^{(n-k)}$ …per ordering!
  - $n!$ orderings
  - k success are alike and $(n-k)$ failures are alike
  - $\frac{n!}{k!(n-k)!}$ orderings of k successes
  - $P\{X=k\} = \binom{n}{k}\theta^k(1-\theta)^{(n-k)}$

Binomial Distribtion

$n = 12, \theta = 0.2$



$n = 12, \theta = 0.6$



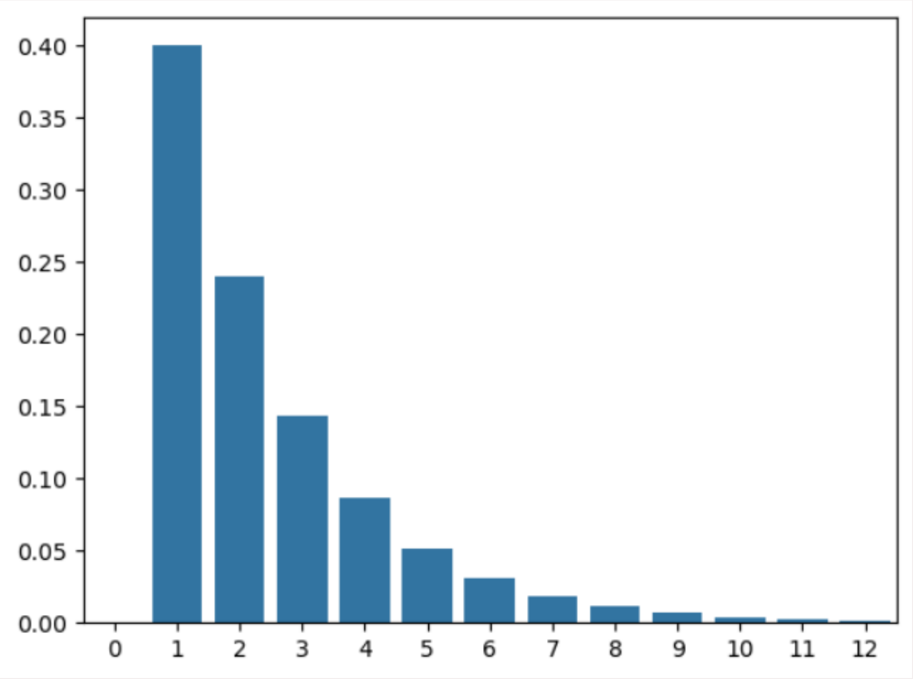
`scipy.stats` 😎

Geometric Distribution

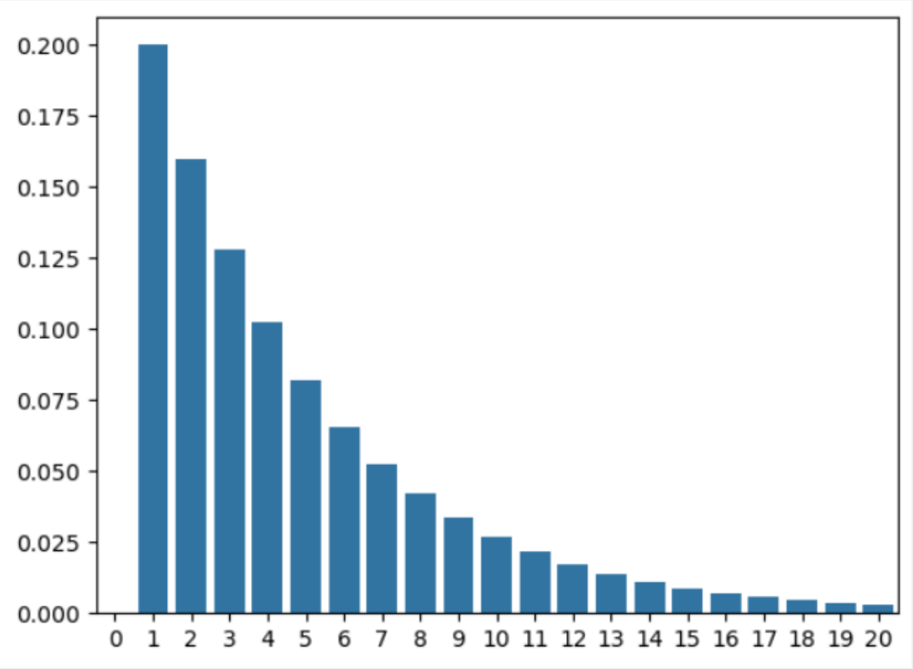
- Perform Bernoulli trials until first success
  - X represents number of failures
  - $P\{X = k\} = \theta(1 - \theta)^{(k)}$ (only one ordering!)

Geometric Distribtion

$\theta = 0.4$



$\theta = 0.2$

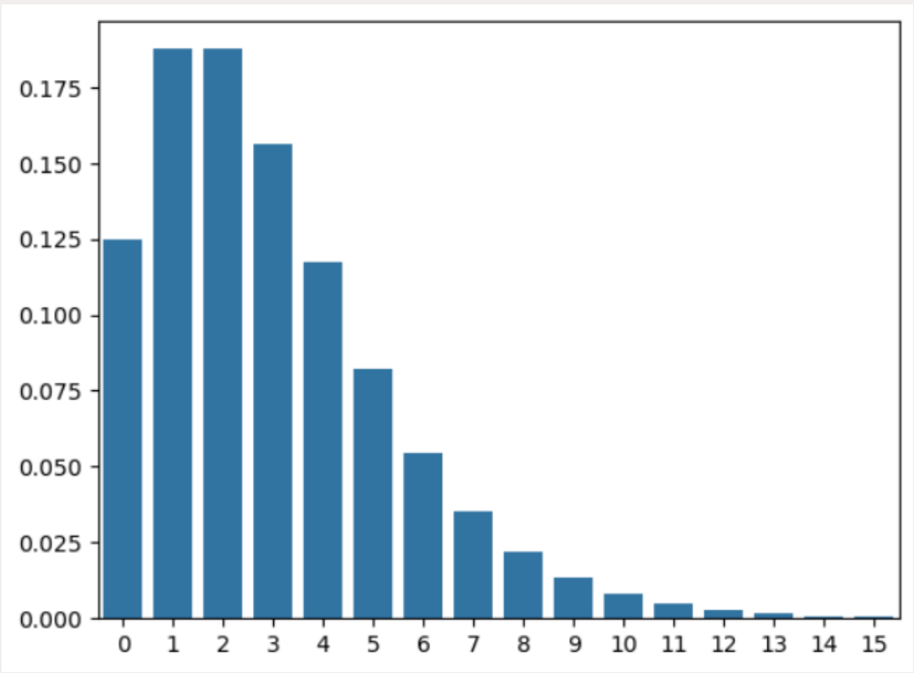


Negative Binomial Distribution

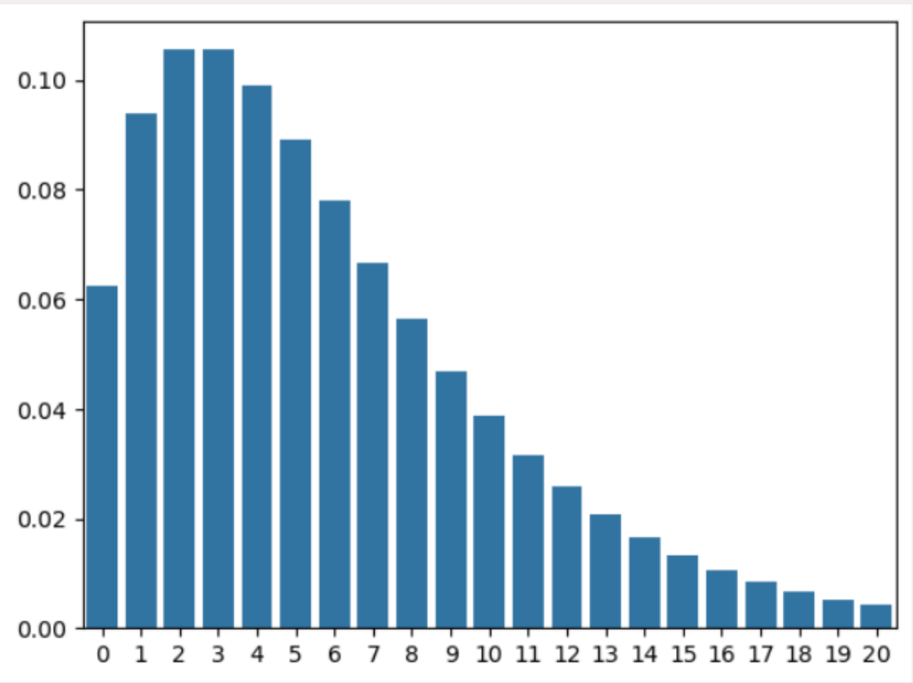
- "General case" of Geometric distribution
- Number of trials until r successes observed
- $P\{X = k\} = \binom{k+r-1}{k}(1-\theta)^k\theta^r$

Negative Binomial Distribution

$r = 3, \theta = 0.5$



$r = 2, \theta = 0.25$



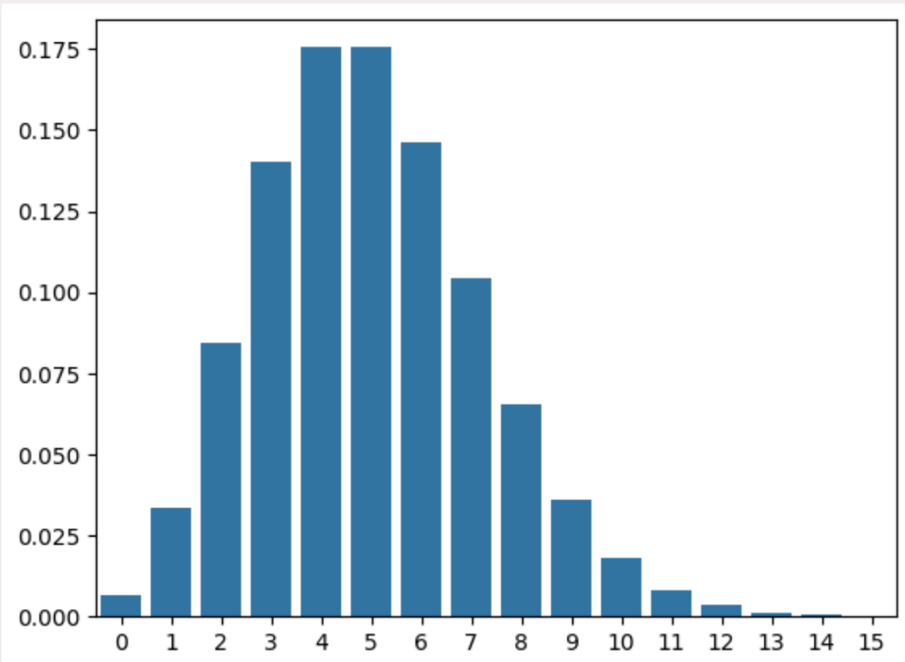
Poisson Distribution

- Events "arrive" *independently* through time
  - People at a bus stop
- Requests to a server
- Number of arrivals per time interval
  - Parameter λ – average number of arrivals

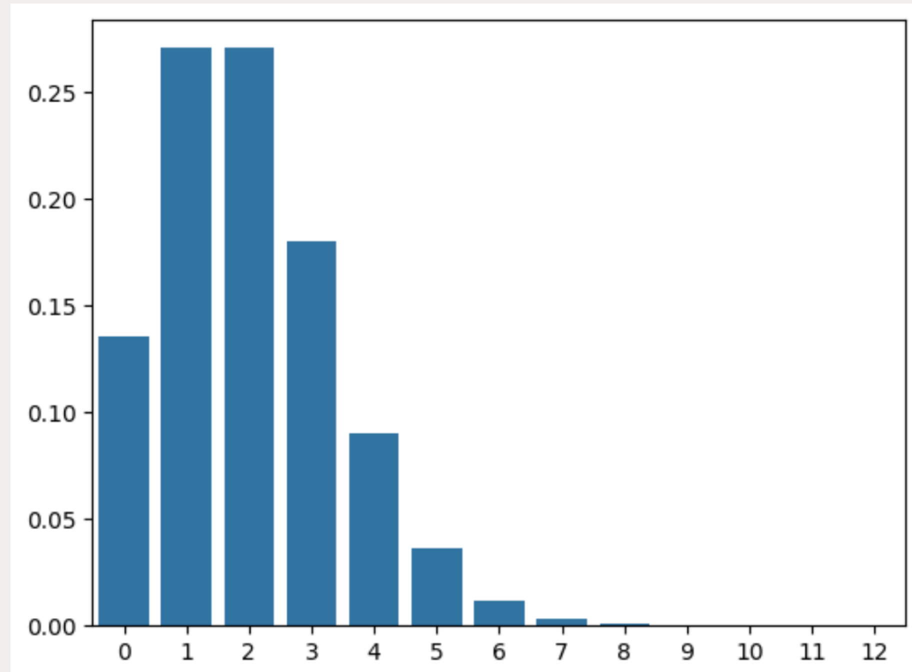
$$P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}$$

Poisson Distribution

$\lambda = 5$



$\lambda = 2$



Continuous Distributions

Continuous vs. Discrete

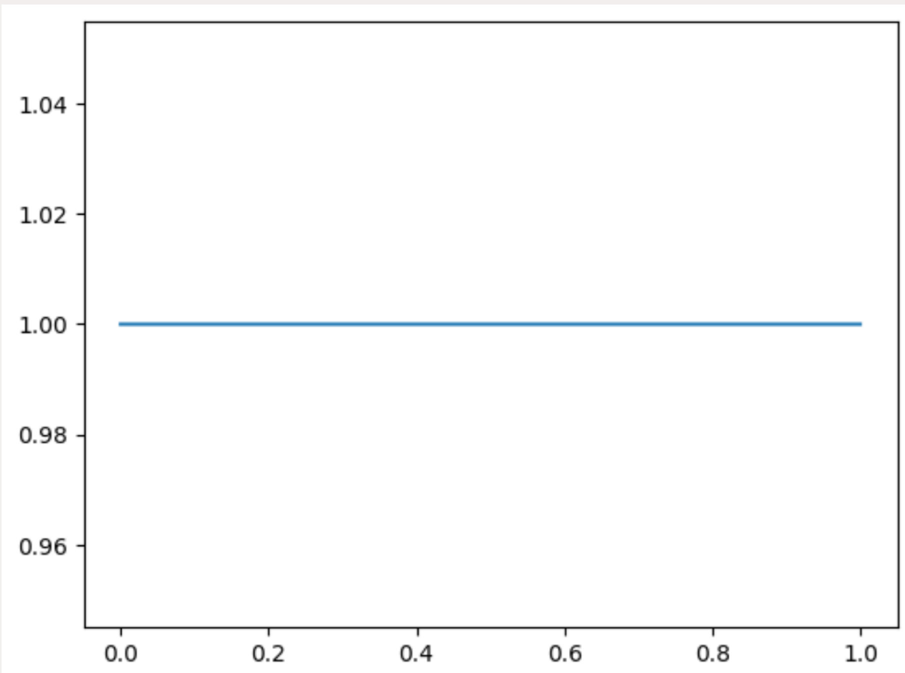
- Discrete:
  - PMF: $p(x)$
  - $E[X] = \sum_i x_i p(x_i)$
- Continuous:
  - PDF: $f(x)$
  - CDF: $P\{X \leq x\} = F(x) = \int_{-\infty}^{x} f(x)\, dx$
  - $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

Uniform Distribution

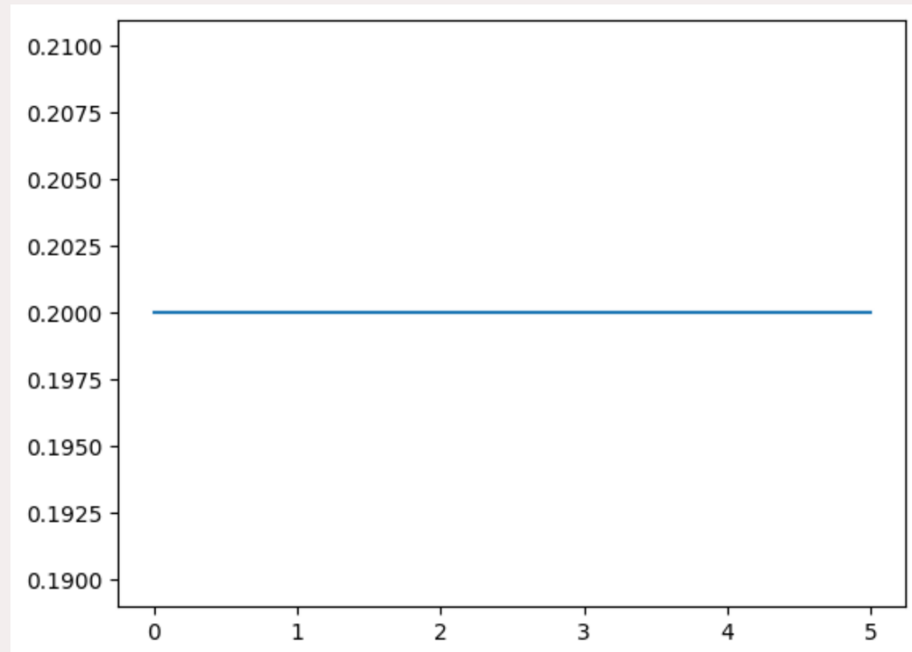
- Takes any value in range with equal probability
  - Range: $[a, b]$
  - Nomenclature: $U(a, b)$
- $U(0, 1)$ is “standard” random variable for modeling

Uniform Distribution

$U(0, 1)$

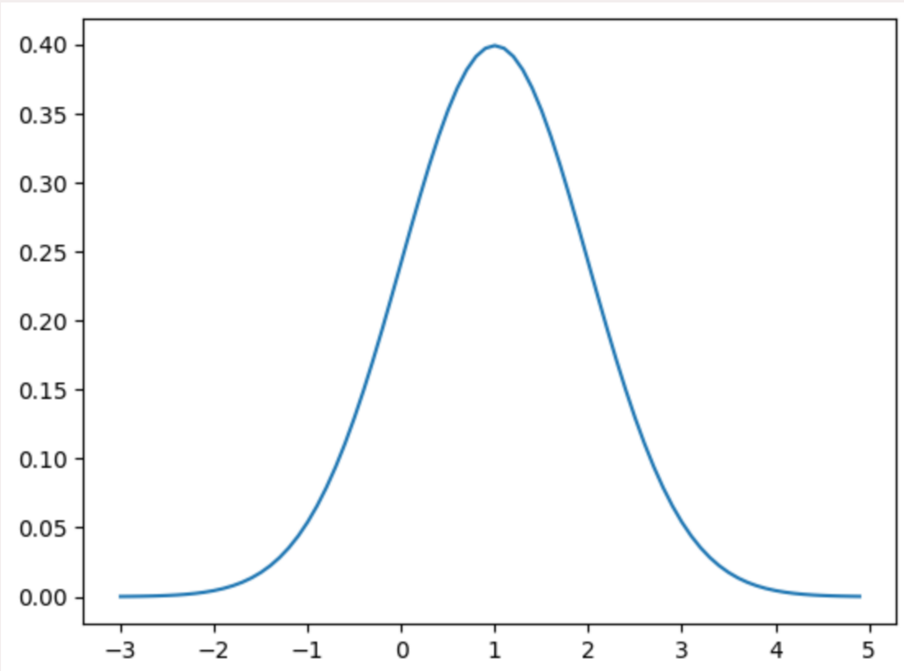


$U(0, 5)$

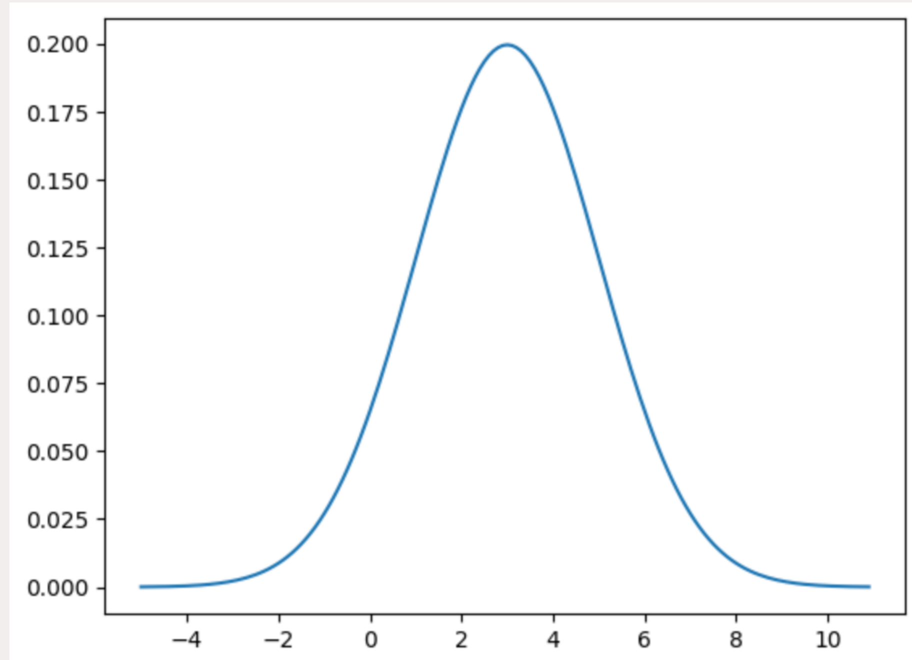


Normal Distribution

$\mu = 1, \sigma^2 = 1$



$\mu = 3, \sigma^2 = 2$



(Remarkably unsatisfying.)

Joint Distributions

- Distribution over multiple variables
  - $P(x, y)$ represents $P\{X = x, Y = y\}$
- Marginal distribution:
  - $P(x) = \sum_y P(x, y)$

Independence

Conditional probability:

$$P(x|y) = \frac{P(x, y)}{P(y)}$$

Bayes' rule:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Conditional Independence

$$P(x|y) = P(x) \rightarrow P(x, y) = P(x)P(y)$$

- Two variables can be conditionally independent…
  - … when conditioned on a third variable

Parameter Space

- n Bernoulli R.V.s
- Fully dependent joint distribution:
  - $2^n - 1$ parameters
- Fully independent joint distribution:
  - n parameters 😌

Notice a theme?

Bayesian Networks

References

- Stuart J. Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach.* 4th Edition, 2020.
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- Ross, *
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